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# Bilinear multipliers of weighted Wiener amalgam spaces and variable exponent Wiener amalgam spaces

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Dedicated to Professor Ravi P Agarwal.

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## Abstract

Let  $\omega_1, \omega_2$  be slowly increasing weight functions, and let  $\omega_3$  be any weight function on  $\mathbb{R}^n$ . Assume that  $m(\xi, \eta)$  is a bounded, measurable function on  $\mathbb{R}^n \times \mathbb{R}^n$ . We define

$$B_m(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) m(\xi, \eta) e^{2\pi i(\xi + \eta, x)} d\xi d\eta$$

for all  $f, g \in C_c^\infty(\mathbb{R}^n)$ . We say that  $m(\xi, \eta)$  is a bilinear multiplier on  $\mathbb{R}^n$  of type  $(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))$  if  $B_m$  is a bounded operator from  $W(L^{p_1}, L^{q_1}_{\omega_1}) \times W(L^{p_2}, L^{q_2}_{\omega_2})$  to  $W(L^{p_3}, L^{q_3}_{\omega_3})$ , where  $1 \leq p_1 \leq q_1 < \infty$ ,  $1 \leq p_2 \leq q_2 < \infty$ ,  $1 < p_3, q_3 \leq \infty$ . We denote by  $\text{BM}(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))$  the vector space of bilinear multipliers of type  $(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))$ . In the first section of this work, we investigate some properties of this space and we give some examples of these bilinear multipliers. In the second section, by using variable exponent Wiener amalgam spaces, we define the bilinear multipliers of type  $(W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3))$  from  $W(L^{p_1(x)}, L^{q_1}_{\omega_1}) \times W(L^{p_2(x)}, L^{q_2}_{\omega_2})$  to  $W(L^{p_3(x)}, L^{q_3}_{\omega_3})$ , where  $p_1^*, p_2^*, p_3^* < \infty$ ,  $p_1(x) \leq q_1$ ,  $p_2(x) \leq q_2$ ,  $1 \leq p_3 \leq \infty$  for all  $p_1(x), p_2(x), p_3(x) \in P(\mathbb{R}^n)$ . We denote by  $\text{BM}(W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3))$  the vector space of bilinear multipliers of type  $(W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3))$ . Similarly, we discuss some properties of this space.

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**Keywords:** bilinear multipliers; weighted Wiener amalgam space; variable exponent Wiener amalgam space

## 1 Introduction

Throughout this paper we will work on  $\mathbb{R}^n$  with Lebesgue measure  $dx$ . We denote by  $C_c^\infty(\mathbb{R}^n)$ ,  $C_c(\mathbb{R}^n)$  and  $S(\mathbb{R}^n)$  the space of infinitely differentiable complex-valued functions with compact support on  $\mathbb{R}^n$ , the space of all continuous, complex-valued functions with compact support on  $\mathbb{R}^n$  and the space of infinitely differentiable complex-valued functions on  $\mathbb{R}^n$  that rapidly decrease at infinity, respectively. Let  $f$  be a complex-valued measurable function on  $\mathbb{R}^n$ . The translation, character and dilation operators  $T_x$ ,  $M_x$  and  $D_s$  are defined by  $T_x f(y) = f(y - x)$ ,  $M_x f(y) = e^{2\pi i(x, y)} f(y)$  and  $D_s^p f(y) = t^{-\frac{n}{p}} f(\frac{y}{t})$ , respectively,

for  $x, y \in \mathbb{R}^n$ ,  $0 < p, t < \infty$ . With this notation out of the way, one has, for  $1 \leq p \leq \infty$  and  $\frac{1}{p} + \frac{1}{p'} = 1$ ,

$$(T_x f)^\wedge(\xi) = M_{-x} \hat{f}(\xi), \quad (M_x f)^\wedge(\xi) = T_x \hat{f}(\xi), \quad (D_t^p f)^\wedge(\xi) = D_{t^{-1}}^{p'} \hat{f}(\xi).$$

For  $1 \leq p \leq \infty$ ,  $L^p(\mathbb{R}^n)$  denotes the usual Lebesgue space. A continuous function  $\omega$  satisfying  $1 \leq \omega(x)$  and  $\omega(x+y) \leq \omega(x)\omega(y)$  for  $x, y \in \mathbb{R}^n$  will be called a weight function on  $\mathbb{R}^n$ . If  $\omega_1(x) \leq \omega_2(x)$  for all  $x \in \mathbb{R}^n$ , we say that  $\omega_1 \leq \omega_2$ . For  $1 \leq p \leq \infty$ , we set

$$L_\omega^p(\mathbb{R}^n) = \{f : f\omega \in L^p(\mathbb{R}^n)\}.$$

It is known that  $L_\omega^p(\mathbb{R}^n)$  is a Banach space under the norm

$$\|f\|_{p,\omega} = \|f\omega\|_p = \left\{ \int_{\mathbb{R}^n} |f(x)\omega(x)|^p dx \right\}^{\frac{1}{p}}, \quad 1 \leq p < \infty$$

or

$$\|f\|_{\infty,\omega} = \|f\omega\|_\infty = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} |f(x)\omega(x)|, \quad p = \infty.$$

The dual of the space  $L_\omega^p(\mathbb{R}^n)$  is the space  $L_{\omega^{-1}}^q(\mathbb{R}^n)$ , where  $\frac{1}{p} + \frac{1}{q} = 1$  and  $\omega^{-1}(x) = \frac{1}{\omega(x)}$ . We say that a weight function  $v_s$  is of polynomial type if  $v_s(x) = (1 + |x|)^s$  for  $s \geq 0$ . Let  $f$  be a measurable function on  $\mathbb{R}^n$ . If there exist  $C > 0$  and  $N \in \mathbb{N}$  such that

$$|f(x)| \leq C(1 + x^2)^N$$

for all  $x \in \mathbb{R}^n$ , then  $f$  is said to be a slowly increasing function [1]. It is easy to see that polynomial-type weight functions are slowly increasing. For  $f \in L^1(\mathbb{R}^n)$ , the Fourier transform of  $f$  is denoted by  $\hat{f}$ . We know that  $\hat{f}$  is a continuous function on  $\mathbb{R}^n$  which vanishes at infinity and it has the inequality  $\|\hat{f}\|_\infty \leq \|f\|_1$ . We denote by  $M(\mathbb{R}^n)$  the space of bounded regular Borel measures, by  $M(\omega)$  the space of  $\mu$  in  $M(\mathbb{R}^n)$  such that

$$\|\mu\|_\omega = \int_{\mathbb{R}^n} \omega d|\mu| < \infty.$$

If  $\mu \in M(\mathbb{R}^n)$ , the Fourier-Stieltjes transform of  $\mu$  is denoted by  $\hat{\mu}$  [2].

The space  $(L^p(\mathbb{R}^n))_{\text{loc}}$  consists of classes of measurable functions  $f$  on  $\mathbb{R}^n$  such that  $f\chi_K \in L^p(\mathbb{R}^n)$  for any compact subset  $K \subset \mathbb{R}^n$ , where  $\chi_K$  is the characteristic function of  $K$ . Let us fix an open set  $Q \subset \mathbb{R}^n$  with compact closure and  $1 \leq p, q \leq \infty$ . The weighted Wiener amalgam space  $W(L^p, L_\omega^q)$  consists of all elements  $f \in (L^p(\mathbb{R}^n))_{\text{loc}}$  such that  $F_f(z) = \|f\chi_{z+Q}\|_p$  belongs to  $L_\omega^q(\mathbb{R}^n)$ ; the norm of  $W(L^p, L_\omega^q)$  is  $\|f\|_{W(L^p, L_\omega^q)} = \|F_f\|_{q,\omega}$  [3–5].

In this paper,  $P(\mathbb{R}^n)$  denotes the family of all measurable functions  $p : \mathbb{R}^n \rightarrow [1, \infty)$ . We put

$$p_* = \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x), \quad p^* = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x).$$

We shall also use the notation

$$\Omega_\infty = \{x \in \mathbb{R}^n : p(x) = \infty\}.$$

The variable exponent Lebesgue spaces (or generalized Lebesgue spaces)  $L^{p(x)}(\mathbb{R}^n)$  are defined as the set of all (equivalence classes) measurable functions  $f$  on  $\mathbb{R}^n$  such that  $\varrho_p(\lambda f) < \infty$  for some  $\lambda > 0$ , equipped with the Luxemburg norm

$$\|f\|_{p(x)} = \inf \left\{ \lambda > 0 : \varrho_p \left( \frac{f}{\lambda} \right) \leq 1 \right\}.$$

If  $p^* < \infty$ , then  $f \in L^{p(x)}(\mathbb{R}^n)$  if  $\varrho_p(f) < \infty$ . The set  $L^{p(x)}(\mathbb{R}^n)$  is a Banach space with the norm  $\|\cdot\|_{p(x)}$ . If  $p(x) = p$  is a constant function, then the norm  $\|\cdot\|_{p(x)}$  coincides with the usual Lebesgue norm  $\|\cdot\|_p$  [6]. The spaces  $L^{p(x)}(\mathbb{R}^n)$  and  $L^p(\mathbb{R}^n)$  have many common properties. A crucial difference between  $L^{p(x)}(\mathbb{R}^n)$  and the classical Lebesgue spaces  $L^p(\mathbb{R}^n)$  is that  $L^{p(x)}(\mathbb{R}^n)$  is not invariant under translation in general. If  $p^* < \infty$ , then  $C_c^\infty(\mathbb{R}^n)$  is dense in  $L^{p(x)}(\mathbb{R}^n)$ . The space  $L^{p(x)}(\mathbb{R}^n)$  is a solid space, that is, if  $f \in L^{p(x)}(\mathbb{R}^n)$  is given and  $g \in L^1_{\text{loc}}(\mathbb{R}^n)$  satisfies  $|g(x)| \leq |f(x)|$  a.e., then  $g \in L^{p(x)}(\mathbb{R}^n)$  and  $\|g\|_{p(x)} \leq \|f\|_{p(x)}$  by [6]. In this paper we will assume that  $p^* < \infty$ .

The space  $(L^{p(x)}(\mathbb{R}^n))_{\text{loc}}$  consists of classes of measurable functions  $f$  on  $\mathbb{R}^n$  such that  $f\chi_K \in L^{p(x)}(\mathbb{R}^n)$  for any compact subset  $K \subset \mathbb{R}^n$ . Let us fix an open set  $Q \subset \mathbb{R}^n$  with compact closure,  $p(x) \in P(\mathbb{R}^n)$  and  $1 \leq q \leq \infty$ . The variable exponent amalgam space  $W(L^{p(x)}, L^q_\omega)$  consists of all elements  $f \in (L^{p(x)}(\mathbb{R}^n))_{\text{loc}}$  such that  $F_f(z) = \|f\chi_{z+Q}\|_{p(x)}$  belongs to  $L^q_\omega(\mathbb{R}^n)$ ; the norm of  $W(L^{p(x)}, L^q_\omega)$  is  $\|f\|_{W(L^{p(x)}, L^q_\omega)} = \|F_f\|_{q,\omega}$  [7].

## 2 The bilinear multipliers space $\text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$

**Lemma 2.1** *Let  $1 \leq p \leq q < \infty$  and  $\omega$  be a slowly increasing weight function. Then  $C_c^\infty(\mathbb{R}^n)$  is dense in the Wiener amalgam space  $W(L^p, L^q_\omega)$ .*

*Proof* Since  $\overline{C_c(\mathbb{R}^n)} = L^q_\omega(\mathbb{R}^n)$  [8], we have  $\overline{C_c(\mathbb{R}^n)} = W(L^p, L^q_\omega)$  by a lemma in [9]. Also we have the inclusion

$$C_c^\infty(\mathbb{R}^n) \subset C_c(\mathbb{R}^n) \subset W(L^p, L^q_\omega).$$

For the proof that  $C_c^\infty(\mathbb{R}^n)$  is dense in  $W(L^p, L^q_\omega)$ , take any  $f \in W(L^p, L^q_\omega)$ . For given  $\varepsilon > 0$ , there exists  $g \in C_c(\mathbb{R}^n)$  such that

$$\|f - g\|_{W(L^p, L^q_\omega)} < \frac{\varepsilon}{2}. \quad (2.1)$$

Also, since  $g \in C_c(\mathbb{R}^n) \subset L^q_\omega(\mathbb{R}^n)$  and  $C_c^\infty(\mathbb{R}^n)$  is dense in  $L^q_\omega(\mathbb{R}^n)$ , by Lemma 2.1 in [10], there exists  $h \in C_c^\infty(\mathbb{R}^n)$  such that

$$\|g - h\|_{q,\omega} < \frac{\varepsilon}{2}.$$

Furthermore, by using the inequality  $p \leq q$ , we write

$$\|g - h\|_{W(L^p, L^q_\omega)} \leq \|g - h\|_{q,\omega} < \frac{\varepsilon}{2} \quad (2.2)$$

(see [11] and [5]). Combining (2.1) and (2.2), we obtain

$$\|f - h\|_{W(L^p, L^q_{\omega})} \leq \|f - g\|_{W(L^p, L^q_{\omega})} + \|h - g\|_{W(L^p, L^q_{\omega})} < \varepsilon.$$

This completes the proof.  $\square$

**Definition 2.1** Let  $1 \leq p_1 \leq q_1 < \infty$ ,  $1 \leq p_2 \leq q_2 < \infty$ ,  $1 < p_3, q_3 \leq \infty$  and  $\omega_1, \omega_2, \omega_3$  be weight functions on  $\mathbb{R}^n$ . Assume that  $\omega_1, \omega_2$  are slowly increasing functions and  $m(\xi, \eta)$  is a bounded, measurable function on  $\mathbb{R}^n \times \mathbb{R}^n$ . Define

$$B_m(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) m(\xi, \eta) e^{2\pi i \langle \xi + \eta, x \rangle} d\xi d\eta$$

for all  $f, g \in C_c^\infty(\mathbb{R}^n)$ .

$m$  is said to be a bilinear multiplier on  $\mathbb{R}^n$  of type  $(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))$  if there exists  $C > 0$  such that

$$\|B_m(f, g)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})} \leq C \|f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})}$$

for all  $f, g \in C_c^\infty(\mathbb{R}^n)$ . That means that  $B_m$  extends to a bounded bilinear operator from  $W(L^{p_1}, L^{q_1}_{\omega_1}) \times W(L^{p_2}, L^{q_2}_{\omega_2})$  to  $W(L^{p_3}, L^{q_3}_{\omega_3})$ .

We denote by  $\text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$  the space of all bilinear multipliers of type  $(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))$  and  $\|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} = \|B_m\|$ .

The following theorem is an example to a bilinear multiplier on  $\mathbb{R}^n$  of type  $(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))$ .

**Theorem 2.1** Let  $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$ ,  $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3}$  and  $\omega_3 \leq \omega_1$ . If  $K \in L^1_{\omega_3}(\mathbb{R}^n)$ , then  $m(\xi, \eta) = \hat{K}(\xi - \eta)$  defines a bilinear multiplier and  $\|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} \leq \|K\|_{1, \omega_3}$ .

*Proof* We know by Theorem 2.1 in [10] that for  $f, g \in C_c^\infty(\mathbb{R}^n)$ ,

$$B_m(f, g)(t) = \int_{\mathbb{R}^n} f(t - y) g(t + y) K(y) dy. \quad (2.3)$$

Also by Proposition 11.4.1 in [5],  $T_y f \in W(L^{p_1}, L^{q_1}_{\omega_1})$ ,  $T_{-y} g \in W(L^{p_2}, L^{q_2}_{\omega_2})$ . So, we write  $F_{T_y f} \in L^{q_1}_{\omega_1}(\mathbb{R}^n)$ ,  $F_{T_{-y} g} \in L^{q_2}_{\omega_2}(\mathbb{R}^n)$ .

Using the Minkowski inequality and the generalized Hölder inequality, we have

$$\begin{aligned} \|B_m(f, g)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})} &= \|B_m(f, g) \chi_{Q+x}\|_{p_3, q_3, \omega_3} \\ &= \left\| \left\{ \int_{\mathbb{R}^n} f(t - y) g(t + y) K(y) dy \right\} \chi_{Q+x} \right\|_{p_3, q_3, \omega_3} \\ &\leq \int_{\mathbb{R}^n} \|f(t - y) g(t + y) \chi_{Q+x}(t)\|_{p_3, q_3, \omega_3} |K(y)| dy \\ &\leq \int_{\mathbb{R}^n} \|f(t - y) \chi_{Q+x}(t)\|_{p_1} \|g(t + y) \chi_{Q+x}(t)\|_{p_2} \|K(y)\|_{q_3, \omega_3} dy \\ &= \int_{\mathbb{R}^n} \|F_{T_y f}(x) \omega_3(x) F_{T_{-y} g}(x)\|_{q_3} |K(y)| dy. \end{aligned} \quad (2.4)$$

Again, by using Proposition 11.4.1 in [5] and the assumption  $\omega_3 \leq \omega_1$ , we write

$$\|F_{T_y f} \omega_3\|_{q_1} \leq \omega_3(y) \|f\|_{W(L^{p_1}, L_{\omega_1}^{q_1})} < \infty. \quad (2.5)$$

From this result, we find  $F_{T_y f} \in L_{\omega_3}^{q_1}(\mathbb{R}^n)$ . Hence by (2.4), (2.5) and the generalized Hölder inequality, we obtain

$$\begin{aligned} \|B_m(f, g)\|_{W(L^{p_3}, L_{\omega_3}^{q_3})} &\leq \int_{\mathbb{R}^n} \|F_{T_y f}(x) \omega_3(x)\|_{q_1} \|F_{T_y g}(x)\|_{q_2} |K(y)| dy \\ &\leq \int_{\mathbb{R}^n} \|f\|_{W(L^{p_1}, L_{\omega_1}^{q_1})} \|g\|_{W(L^{p_2}, L_{\omega_2}^{q_2})} |K(y)| \omega_3(y) dy \\ &= \|f\|_{W(L^{p_1}, L_{\omega_1}^{q_1})} \|g\|_{W(L^{p_2}, L_{\omega_2}^{q_2})} \|K\|_{1, \omega_3} \\ &= C \|f\|_{W(L^{p_1}, L_{\omega_1}^{q_1})} \|g\|_{W(L^{p_2}, L_{\omega_2}^{q_2})}, \end{aligned} \quad (2.6)$$

where  $C = \|K\|_{1, \omega_3}$ . Then  $m(\xi, \eta) = \hat{K}(\xi - \eta)$  defines a bilinear multiplier. Finally, using (2.6), we obtain

$$\begin{aligned} \|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} &= \sup_{\|f\|_{W(L^{p_1}, L_{\omega_1}^{q_1})} \leq 1, \|g\|_{W(L^{p_2}, L_{\omega_2}^{q_2})} \leq 1} \frac{\|B_m(f, g)\|_{W(L^{p_3}, L_{\omega_3}^{q_3})}}{\|f\|_{W(L^{p_1}, L_{\omega_1}^{q_1})} \|g\|_{W(L^{p_2}, L_{\omega_2}^{q_2})}} \\ &\leq \|K\|_{1, \omega_3}. \end{aligned} \quad \square$$

**Definition 2.2** Let  $1 \leq p_1 \leq p_2 < \infty$ ,  $1 \leq q_1 \leq q_2 < \infty$ ,  $1 < p_3, q_3 \leq \infty$  and  $\omega_1, \omega_2, \omega_3$  be weight functions on  $\mathbb{R}^n$ . Suppose that  $\omega_1, \omega_2$  are slowly increasing functions. We denote by  $\tilde{M}[(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$  the space of measurable functions  $M: \mathbb{R}^n \rightarrow \mathbb{C}$  such that  $m(\xi, \eta) = M(\xi - \eta) \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ , that is to say,

$$B_M(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) M(\xi - \eta) e^{2\pi i(\xi + \eta) \cdot x} d\xi d\eta$$

extends to a bounded bilinear map from  $W(L^{p_1}, L_{\omega_1}^{q_1}) \times W(L^{p_2}, L_{\omega_2}^{q_2})$  to  $W(L^{p_3}, L_{\omega_3}^{q_3})$ . We denote  $\|M\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} = \|B_M\|$ .

Let  $\omega$  be a weight function. The continuous function  $\omega^{-1}$  cannot be a weight function. But the following lemma can be proved easily by using the technique of the proof of Lemma 2.1.

**Lemma 2.2** Let  $1 \leq p \leq q < \infty$  and  $\omega$  be a slowly increasing continuous weight function. Then  $C_c^\infty(\mathbb{R}^n)$  is dense in  $W(L^p, L_{\omega^{-1}}^q)$  Wiener amalgam space.

**Theorem 2.2** Let  $\frac{1}{p_3} + \frac{1}{p'_3} = 1$ ,  $\frac{1}{q_3} + \frac{1}{q'_3} = 1$ ,  $q'_3 \geq p'_3 \geq 1$  and  $\omega_3$  be a continuous, symmetric slowly increasing weight function. Then  $m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$  if and only if there exists  $C > 0$  such that

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\xi + \eta) m(\xi, \eta) d\xi d\eta \right| \leq C \|f\|_{W(L^{p_1}, L_{\omega_1}^{q_1})} \|g\|_{W(L^{p_2}, L_{\omega_2}^{q_2})} \|h\|_{W(L^{p'_3}, L_{\omega_3^{-1}}^{q'_3})}$$

for all  $f, g, h \in C_c^\infty(\mathbb{R}^n)$ .

*Proof* We assume that  $m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ . By Theorem 2.2 in [10], we write, for all  $f, g, h \in C_c^\infty(\mathbb{R}^n)$ ,

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\xi + \eta) m(\xi, \eta) d\xi d\eta \right| &= \left| \int_{\mathbb{R}^n} h(y) \tilde{B}_m(f, g)(y) dy \right| \\ &\leq \int_{\mathbb{R}^n} |h(y)| |\tilde{B}_m(f, g)(y)| dy, \end{aligned} \quad (2.7)$$

where  $\tilde{B}_m(f, g)(y) = B_m(f, g)(-y)$ . If we set  $-t = u$ , we have

$$\begin{aligned} \|\tilde{B}_m(f, g)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})} &= \|F_{\tilde{B}_m(f, g)}^Q\|_{q_3, \omega_3} = \|B_m(f, g)(u) \chi_{Q+x}(-u)\|_{p_3} \|_{q_3, \omega_3} \\ &= \|B_m(f, g)(u) \chi_{-Q-x}(u)\|_{p_3} \|_{q_3, \omega_3} = \|F_{B_m(f, g)}^{-Q}(-x)\|_{q_3, \omega_3}. \end{aligned} \quad (2.8)$$

Since  $\omega_3$  is a symmetric weight function, if we set  $-x = y$ , we have

$$\|\tilde{B}_m(f, g)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})} = \|F_{B_m(f, g)}^{-Q}(y)\|_{q_3, \omega_3}. \quad (2.9)$$

We know from [3] and [5] that the definition of  $W(L^{p_3}, L^{q_3}_{\omega_3})$  is independent of the choice of  $Q$ . Then there exists  $C > 0$  such that

$$\|F_{B_m(f, g)}^{-Q}(y)\|_{q_3, \omega_3} \leq C_1 \|F_{B_m(f, g)}^Q(y)\|_{q_3, \omega_3}. \quad (2.10)$$

Hence, by (2.9) and (2.10), we have

$$\|\tilde{B}_m(f, g)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})} \leq C_1 \|F_{B_m(f, g)}^Q(y)\|_{q_3, \omega_3} = C_1 \|B_m(f, g)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})}. \quad (2.11)$$

Since from the assumption  $m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$  the right-hand side of (2.11) is finite, thus  $\tilde{B}_m(f, g) \in W(L^{p_3}, L^{q_3}_{\omega_3})$ . On the other hand, since  $m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ , there exists  $C_2 > 0$  such that

$$\|B_m(f, g)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})} \leq C_2 \|f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})}. \quad (2.12)$$

Combining (2.11) and (2.12), we have

$$\|\tilde{B}_m(f, g)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})} \leq C_1 C_2 \|f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})}. \quad (2.13)$$

If we apply the Hölder inequality to the right-hand side of inequality (2.7) and use inequality (2.13), we obtain

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\xi + \eta) m(\xi, \eta) d\xi d\eta \right| &\leq \|\tilde{B}_m(f, g)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})} \|h\|_{W(L^{p'_3}, L^{q'_3}_{\omega_3^{-1}})} \\ &\leq C_1 C_2 \|f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})} \|h\|_{W(L^{p'_3}, L^{q'_3}_{\omega_3^{-1}})}. \end{aligned}$$

For the proof of converse, assume that there exists a constant  $C > 0$  such that

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\xi + \eta) m(\xi, \eta) d\xi d\eta \right| \leq C \|f\|_{W(L^{p_1}, L_{\omega_1}^{q_1})} \|g\|_{W(L^{p_2}, L_{\omega_2}^{q_2})} \|h\|_{W(L^{p'_3}, L_{\omega_3}^{q'_3})} \quad (2.14)$$

for all  $f, g, h \in C_c^\infty(\mathbb{R}^n)$ . From the assumption and (2.14), we write

$$\left| \int_{\mathbb{R}^n} h(y) \tilde{B}_m(f, g)(y) dy \right| \leq C \|f\|_{W(L^{p_1}, L_{\omega_1}^{q_1})} \|g\|_{W(L^{p_2}, L_{\omega_2}^{q_2})} \|h\|_{W(L^{p'_3}, L_{\omega_3}^{q'_3})}. \quad (2.15)$$

Define a function  $\ell$  from  $C_c^\infty(\mathbb{R}^n) \subset W(L^{p'_3}, L_{\omega_3}^{q'_3})$  to  $\mathbb{C}$  such that

$$\ell(h) = \int_{\mathbb{R}^n} h(y) \tilde{B}_m(f, g)(y) dy.$$

$\ell$  is linear and bounded by (2.15). Also, since  $q'_3 \geq p'_3 \geq 1$ , we have  $\overline{C_c^\infty(\mathbb{R}^n)} = W(L^{p'_3}, L_{\omega_3}^{q'_3})$  by Lemma 2.2. Thus  $\ell$  extends to a bounded function from  $W(L^{p'_3}, L_{\omega_3}^{q'_3})$  to  $\mathbb{C}$ . Then  $\ell \in (W(L^{p'_3}, L_{\omega_3}^{q'_3}))^* = W(L^{p_3}, L_{\omega_3}^{q_3})$ . Again, since the definition of  $W(L^{p_3}, L_{\omega_3}^{q_3})$  is independent of the choice of  $Q$ , there exists  $C_3 > 0$  such that

$$\|B_m(f, g)\|_{W(L^{p_3}, L_{\omega_3}^{q_3})} \leq C_3 \|\tilde{B}_m(f, g)\|_{W(L^{p_3}, L_{\omega_3}^{q_3})}. \quad (2.16)$$

Combining (2.15) and (2.16), we obtain

$$\begin{aligned} \|B_m(f, g)\|_{W(L^{p_3}, L_{\omega_3}^{q_3})} &\leq C_3 \|\tilde{B}_m(f, g)\|_{W(L^{p_3}, L_{\omega_3}^{q_3})} \\ &= C_3 \|\ell\| = C_3 \sup_{\|h\|_{W(L^{p'_3}, L_{\omega_3}^{q'_3})} \leq 1} \frac{|\ell(h)|}{\|h\|_{W(L^{p'_3}, L_{\omega_3}^{q'_3})}} \\ &\leq C_3 C \|f\|_{W(L^{p_1}, L_{\omega_1}^{q_1})} \|g\|_{W(L^{p_2}, L_{\omega_2}^{q_2})}. \end{aligned}$$

This completes proof.  $\square$

The following theorem is a generalization of Theorem 2.1.

**Theorem 2.3** Let  $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$ ,  $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3}$ ,  $\omega_3 \leq \omega_1$  and  $v(x) = C(1 + x^2)^N$ ,  $C \geq 0$ ,  $N \in \mathbb{N}$  be a weight function. If  $\mu \in M(v)$  and  $m(\xi, \eta) = \hat{\mu}(\alpha\xi + \beta\eta)$  for  $\alpha, \beta \in \mathbb{R}$ , then  $m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ . Moreover,

$$\begin{aligned} \|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} &\leq \|\mu\|_v \quad \text{if } |\alpha| \leq 1, \\ \|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} &\leq |\alpha|^{2N} \|\mu\|_v \quad \text{if } |\alpha| > 1. \end{aligned}$$

*Proof* Let  $f, g \in C_c^\infty(\mathbb{R}^n)$ . By Theorem 2.3 in [10], we have

$$B_m(f, g)(t) = \int_{\mathbb{R}^n} f(t - \alpha y) g(t - \beta y) d\mu(y). \quad (2.17)$$

Also by [5] we write the inequalities

$$\|T_{\alpha y}f\|_{W(L^{p_1}, L_{\omega_1}^{q_1})} \leq \omega_1(\alpha y) \|f\|_{W(L^{p_1}, L_{\omega_1}^{q_1})} \quad (2.18)$$

and

$$\|T_{\beta y}g\|_{W(L^{p_2}, L_{\omega_2}^{q_2})} \leq \omega_2(\alpha y) \|g\|_{W(L^{p_2}, L_{\omega_2}^{q_2})}. \quad (2.19)$$

From these inequalities, we have  $F_{T_{\alpha y}f} \in L_{\omega_1}^{q_1}(\mathbb{R}^n)$  and  $F_{T_{\beta y}g} \in L_{\omega_2}^{q_2}(\mathbb{R}^n)$ . If we use the inequality  $\omega_3 \leq \omega_1$  and set  $x - \alpha t = u$ , we obtain

$$\|F_{T_{\alpha y}f}\omega_3\|_{q_1} \leq \|F_{T_{\alpha y}f}\omega_1\|_{q_1} \leq \omega_1(\alpha y) \|f\|_{W(L^{p_1}, L_{\omega_1}^{q_1})}, \quad (2.20)$$

and hence  $F_{T_{\alpha y}f}\omega_3 \in L^{q_1}(\mathbb{R}^n)$ . Then by (2.17), (2.18), (2.19), (2.20) and the Hölder inequality, we have

$$\begin{aligned} \|B_m(f, g)\|_{W(L^{p_3}, L_{\omega_3}^{q_3})} &\leq \left\| \int_{\mathbb{R}^n} \|f(t - \alpha y)g(t - \beta y)\chi_{Q+x}(t)\|_{p_3} d|\mu|(y) \right\|_{q_3, \omega_3} \\ &\leq \left\| \int_{\mathbb{R}^n} \|f(t - \alpha y)\chi_{Q+x}(t)\|_{p_1} \|g(t - \beta y)\chi_{Q+x}(t)\|_{p_2} d|\mu|(y) \right\|_{q_3, \omega_3} \\ &\leq \int_{\mathbb{R}^n} \|F_{T_{\alpha y}f}(x)F_{T_{\beta y}g}(x)\|_{q_3, \omega_3} d|\mu|(y) \\ &\leq \int_{\mathbb{R}^n} \|F_{T_{\alpha y}f}(x)\omega_3(x)\|_{q_1} \|F_{T_{\beta y}g}(x)\|_{q_2} d|\mu|(y) \\ &\leq \int_{\mathbb{R}^n} \omega_1(\alpha y) \|f\|_{W(L^{p_1}, L_{\omega_1}^{q_1})} \|g\|_{W(L^{p_2}, L_{\omega_2}^{q_2})} d|\mu|(y) \\ &\leq \|f\|_{W(L^{p_1}, L_{\omega_1}^{q_1})} \|g\|_{W(L^{p_2}, L_{\omega_2}^{q_2})} \int_{\mathbb{R}^n} \omega_1(\alpha y) d|\mu|(y). \end{aligned} \quad (2.21)$$

Now, suppose that  $\alpha \leq 1$ . Since  $\omega_1$  is a slowly increasing weight function, there exist  $C \geq 0$  and  $N \in \mathbb{N}$  such that

$$\omega_1(x) \leq C(1 + x^2)^N = v(x).$$

Then

$$\int_{\mathbb{R}^n} \omega_1(\alpha y) d|\mu|(y) \leq \int_{\mathbb{R}^n} C(1 + \alpha^2 y^2)^N d|\mu|(y) \leq \int_{\mathbb{R}^n} C(1 + y^2)^N d|\mu|(y) = \|\mu\|_v.$$

Hence by (2.21)

$$\|B_m(f, g)\|_{W(L^{p_3}, L_{\omega_3}^{q_3})} \leq \|f\|_{W(L^{p_1}, L_{\omega_1}^{q_1})} \|g\|_{W(L^{p_2}, L_{\omega_2}^{q_2})} \|\mu\|_v. \quad (2.22)$$

Thus  $m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ , and by (2.22) we obtain

$$\begin{aligned} \|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} &= \sup_{\|f\|_{W(L^{p_1}, L_{\omega_1}^{q_1})} \leq 1, \|g\|_{W(L^{p_2}, L_{\omega_2}^{q_2})} \leq 1} \frac{\|B_m(f, g)\|_{W(L^{p_3}, L_{\omega_3}^{q_3})}}{\|f\|_{W(L^{p_1}, L_{\omega_1}^{q_1})} \|g\|_{W(L^{p_2}, L_{\omega_2}^{q_2})}} \\ &\leq \|\mu\|_v. \end{aligned}$$



Similarly, if  $\alpha > 1$ , then

$$\int_{\mathbb{R}^n} \omega_1(\alpha y) d|\mu|(y) \leq \int_{\mathbb{R}^n} C(\alpha^2 + \alpha^2 y^2)^N d|\mu|(y) = \alpha^{2N} \int_{\mathbb{R}^n} v(y) d|\mu|(y) = \alpha^{2N} \|\mu\|_v.$$

Therefore by (2.21) we have

$$\|B_m(f, g)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})} \leq \alpha^{2N} \|f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})} \|\mu\|_v. \quad (2.23)$$

Hence, we obtain  $m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ , and by (2.23)

$$\begin{aligned} \|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} &= \sup_{\|f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \leq 1, \|g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})} \leq 1} \frac{\|B_m(f, g)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})}}{\|f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})}} \\ &\leq \alpha^{2N} \|\mu\|_v. \quad \square \end{aligned}$$

Now, we will give some properties of the space  $\text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ .

**Theorem 2.4** Let  $m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ .

(a)  $T_{(\xi_0, \eta_0)} m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$  for each  $(\xi_0, \eta_0) \in \mathbb{R}^{2n}$  and

$$\|T_{(\xi_0, \eta_0)} m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} = \|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))}.$$

(b)  $M_{(\xi_0, \eta_0)} m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$  for each  $(\xi_0, \eta_0) \in \mathbb{R}^{2n}$  and

$$\begin{aligned} \|M_{(\xi_0, \eta_0)} m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} \\ \leq \omega_1(-\xi_0) \omega_2(-\eta_0) \|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))}. \end{aligned}$$

*Proof* (a) Let  $f, g \in C_c^\infty(\mathbb{R}^n)$ . From Theorem 2.4 in [10], we write the equality

$$B_{T_{(\xi_0, \eta_0)} m}(f, g)(x) = e^{2\pi i(\xi_0 + \eta_0, x)} B_m(M_{-\xi_0} f, M_{-\eta_0} g)(x). \quad (2.24)$$

Also the equalities  $\|M_{-\xi_0} f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} = \|f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})}$  and  $\|M_{-\eta_0} g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})} = \|g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})}$  are satisfied. Then, using equality (2.24) and the assumption  $m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ , we have

$$\begin{aligned} \|B_{T_{(\xi_0, \eta_0)} m}(f, g)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})} &= \|e^{2\pi i(\xi_0 + \eta_0, x)} B_m(M_{-\xi_0} f, M_{-\eta_0} g)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})} \\ &= \|B_m(M_{-\xi_0} f, M_{-\eta_0} g)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})} \\ &\leq C \|f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})} \end{aligned}$$

for some  $C > 0$ . Thus  $T_{(\xi_0, \eta_0)} m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ . Moreover, we obtain

$$\begin{aligned} \|T_{(\xi_0, \eta_0)} m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} \\ = \|B_{T_{(\xi_0, \eta_0)} m}\| = \sup_{\|f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \leq 1, \|g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})} \leq 1} \frac{\|B_{T_{(\xi_0, \eta_0)} m}(f, g)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})}}{\|f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})}} \end{aligned}$$

$$\begin{aligned}
 &= \sup_{\|M_{-\xi_0}f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \leq 1, \|M_{-\eta_0}g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})} \leq 1} \frac{\|B_{T_{(\xi_0, \eta_0)}m}(M_{-\xi_0}f, M_{-\eta_0}g)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})}}{\|M_{-\xi_0}f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \|M_{-\eta_0}g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})}} \\
 &= \|B_m\| = \|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))}.
 \end{aligned}$$

(b) For any  $f, g \in C_c^\infty(\mathbb{R}^n)$ , we write

$$B_{M_{(\xi_0, \eta_0)}m}(f, g)(x) = B_m(T_{-\xi_0}f, T_{-\eta_0}g)(x) \quad (2.25)$$

by Theorem 2.4 in [10]. Also, the inequalities  $\|T_{-\xi_0}f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \leq \omega_1(-\xi_0)\|f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})}$  and  $\|T_{-\eta_0}g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})} \leq \omega_2(-\eta_0)\|g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})}$  are satisfied [5]. Since  $m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ , by (2.25) we have

$$\begin{aligned}
 \|B_{M_{(\xi_0, \eta_0)}m}(f, g)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})} &= \|B_m(T_{-\xi_0}f, T_{-\eta_0}g)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})} \\
 &\leq \|B_m\| \|T_{-\xi_0}f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \|T_{-\eta_0}g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})} \\
 &\leq \omega_1(-\xi_0)\omega_2(-\eta_0)\|B_m\| \|f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})} \quad (2.26)
 \end{aligned}$$

and hence  $M_{(\xi_0, \eta_0)}m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ . So, by (2.26) we obtain

$$\begin{aligned}
 &\|M_{(\xi_0, \eta_0)}m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} \\
 &= \sup_{\|f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \leq 1, \|g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})} \leq 1} \frac{\|B_{M_{(\xi_0, \eta_0)}m}(f, g)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})}}{\|f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})}} \\
 &\leq \omega_1(-\xi_0)\omega_2(-\eta_0)\|B_m\| = \omega_1(-\xi_0)\omega_2(-\eta_0)\|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))}. \quad \square
 \end{aligned}$$

**Lemma 2.3** If  $\omega$  is a slowly increasing weight function such that  $\omega(x) \leq C_1(1+x^2)^N = v(x)$  and  $f \in W(L^p, L^q_\omega)$ , then  $D_y^p f \in W(L^p, L^q_v)$ . Moreover,

$$\begin{aligned}
 \|D_y^p f\|_{W(L^p, L^q_\omega)} &\leq C\|f\|_{W(L^p, L^q_v)} \quad \text{if } y \leq 1, \\
 \|D_y^p f\|_{W(L^p, L^q_\omega)} &< Cy^{\frac{n}{p}+2N}\|f\|_{W(L^p, L^q_v)} \quad \text{if } y > 1
 \end{aligned}$$

for some  $C > 0$ .

*Proof* Take any  $f \in W(L^p, L^q_\omega)$ . If we get  $\frac{t}{y} = u$ , we obtain

$$\begin{aligned}
 \|D_y^p f\|_{W(L^p, L^q_\omega)} &= \left\| \left\{ \int_{\mathbb{R}^n} |D_y^p f(t) \chi_{Q+x}(t)|^p dt \right\}^{\frac{1}{p}} \right\|_{q, \omega} \\
 &= \left\| \left\{ \int_{\mathbb{R}^n} \left| y^{-\frac{n}{p}} f\left(\frac{t}{y}\right) \right|^p \chi_{Q+x}(t) dt \right\}^{\frac{1}{p}} \right\|_{q, \omega} \\
 &= \left\| \left\{ \int_{\mathbb{R}^n} |f(u)|^p \chi_{y^{-1}Q+y^{-1}x}(u) du \right\}^{\frac{1}{p}} \right\|_{q, \omega} \\
 &= \|F_f^{y^{-1}Q}(y^{-1}x)\|_{q, \omega}. \quad (2.27)
 \end{aligned}$$

Again, if we say  $y^{-1}x = s$  and use  $\omega$  to be slowly increasing, then there exist  $C_1 > 0$  and  $N \in \mathbb{N}$  such that

$$\begin{aligned} \|D_y^p f\|_{W(L^p, L_\omega^q)} &= \left\{ \int_{\mathbb{R}^n} |F_f^{y^{-1}Q}(y^{-1}x)|^q \omega(x)^q dx \right\}^{\frac{1}{q}} \\ &= y^{\frac{n}{q}} \left\{ \int_{\mathbb{R}^n} |F_f^{y^{-1}Q}(s)|^q \omega(yx)^q ds \right\}^{\frac{1}{q}} \\ &\leq y^{\frac{n}{q}} \left\{ \int_{\mathbb{R}^n} |F_f^{y^{-1}Q}(s)|^q (C_1(1+y^2s^2)^N)^q ds \right\}^{\frac{1}{q}} \end{aligned} \quad (2.28)$$

by equation (2.27).

Let  $y \leq 1$ . Using inequality (2.28), we have

$$\|D_y^p f\|_{W(L^p, L_\omega^q)} \leq \left\{ \int_{\mathbb{R}^n} |F_f^{y^{-1}Q}(s)|^q (C_1(1+s^2)^N)^q ds \right\}^{\frac{1}{q}} = \|F_f^{y^{-1}Q}\|_{q,v}. \quad (2.29)$$

Since  $y^{-1}Q$  is a compact set and the definition of  $W(L^p, L_v^q)$  is independent of the choice of a compact set  $Q$ , then there exists  $C > 0$  such that

$$\|F_f^{y^{-1}Q}\|_{q,v} \leq C \|F_f^Q\|_{q,v} \quad (2.30)$$

by [3, 5]. Then by (2.29) we write

$$\|D_y^p f\|_{W(L^p, L_\omega^q)} \leq \|F_f^{y^{-1}Q}\|_{q,v} \leq C \|F_f^Q\|_{q,v} = C \|f\|_{W(L^p, L_v^q)}.$$

Thus we have  $D_y^p f \in W(L^p, L_v^q)$ .

Now, assume that  $y > 1$ . Similarly, by (2.28) and (2.30), we get

$$\begin{aligned} \|D_y^p f\|_{W(L^p, L_\omega^q)} &< y^{\frac{n}{q}} \left\{ \int_{\mathbb{R}^n} |F_f^{y^{-1}Q}(s)|^q (C_1(y^2 + y^2s^2)^N)^q ds \right\}^{\frac{1}{q}} \\ &= y^{\frac{n}{q}+2N} \left\{ \int_{\mathbb{R}^n} |F_f^{y^{-1}Q}(s)|^q v(s)^q ds \right\}^{\frac{1}{q}} \\ &\leq y^{\frac{n}{q}+2N} \|f\|_{W(L^p, L_v^q)}. \end{aligned}$$

Hence  $D_y^p f \in W(L^p, L_v^q)$ . □

**Theorem 2.5** Let  $v_i(x) = C_i(1+x^2)^{N_i}$ ,  $C_i > 0$ ,  $N_i > 0$  for  $i = 1, 2, 3$ , and let  $\omega_3$  be a slowly increasing weight function. If  $\frac{2}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3}$ ,  $0 < y < \infty$  and  $m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ , then  $D_y^q m \in \text{BM}[W(p_1, q_1, v_1; p_2, q_2, v_2; p_3, q_3, \omega_3)]$ . Moreover, then

$$\begin{aligned} \|D_y^q m\|_{(W(p_1, q_1, v_1; p_2, q_2, v_2; p_3, q_3, \omega_3))} &\leq Cy^{-\frac{n}{q_3}-2N_3} \|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, v_3))} \quad \text{if } y \leq 1, \\ \|D_y^q m\|_{(W(p_1, q_1, v_1; p_2, q_2, v_2; p_3, q_3, \omega_3))} &< Cy^{\frac{n}{q_1} + \frac{n}{q_2} + 2N_1 + 2N_2} \|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, v_3))} \quad \text{if } y > 1. \end{aligned}$$

*Proof* Let  $f \in W(L^{p_1}, L_{\omega_1}^{q_1})$  and  $g \in W(L^{p_2}, L_{\omega_2}^{q_2})$  be given. From Lemma 2.3, we have  $D_y^{p_1} f \in W(L^{p_1}, L_{\omega_1}^{q_1})$  and  $D_y^{p_2} g \in W(L^{p_2}, L_{\omega_2}^{q_2})$ . Also it is known by Theorem 2.5 in [10] that

$$B_{D_y^q m}(f, g)(y) = D_{y^{-1}}^{p_3} B_m(D_y^{p_1} f, D_y^{p_2} g)(y).$$

If we use this equality, we write

$$\begin{aligned} \|B_{D_y^q m}(f, g)\|_{W(L^{p_3}, L_{\omega_3}^{q_3})} &= \left\| \left\{ \int_{\mathbb{R}^n} |D_{y^{-1}}^{p_3} B_m(D_y^{p_1} f, D_y^{p_2} g)(t) \chi_{Q+x}(t)|^{p_3} dt \right\}^{\frac{1}{p_3}} \right\|_{q_3, \omega_3} \\ &= \left\| \left\{ \int_{\mathbb{R}^n} |y^{\frac{n}{p_3}} B_m(D_y^{p_1} f, D_y^{p_2} g)\left(\frac{t}{y^{-1}}\right) \chi_{Q+x}(t)|^{p_3} dt \right\}^{\frac{1}{p_3}} \right\|_{q_3, \omega_3} \\ &= \left\| \left\{ \int_{\mathbb{R}^n} |y^n B_m(D_y^{p_1} f, D_y^{p_2} g)(ty) \chi_{Q+x}(t)|^{p_3} dt \right\}^{\frac{1}{p_3}} \right\|_{q_3, \omega_3}. \end{aligned}$$

If we say  $yt = u$  in the last equality, we have

$$\begin{aligned} \|B_{D_y^q m}(f, g)\|_{W(L^{p_3}, L_{\omega_3}^{q_3})} &= \left\| \left\{ \int_{\mathbb{R}^n} |B_m(D_y^{p_1} f, D_y^{p_2} g)(u) \chi_{yQ+yx}(u)|^{p_3} dt \right\}^{\frac{1}{p_3}} \right\|_{q_3, \omega_3} \\ &= \|F_{B_m(D_y^{p_1} f, D_y^{p_2} g)}^{yQ}(yx)\|_{q_3, \omega_3}. \end{aligned} \quad (2.31)$$

On the other hand, since  $\omega_3$  is a slowly increasing weight function, there exist  $C_3 > 0$ ,  $N_3 > 0$  such that  $\omega_3(x) \leq C_3(1+x^2)^{N_3} = v_3(x)$ . If we make the substitution  $yx = s$  in equality (2.31), we obtain

$$\begin{aligned} \|B_{D_y^q m}(f, g)\|_{W(L^{p_3}, L_{\omega_3}^{q_3})} &= \|F_{B_m(D_y^{p_1} f, D_y^{p_2} g)}^{yQ}(yx)\|_{q_3, \omega_3} \\ &= \left\{ \int_{\mathbb{R}^n} |F_{B_m(D_y^{p_1} f, D_y^{p_2} g)}^{yQ}(s)|^{q_3} \omega_3(y^{-1}s)^{q_3} y^{-n} ds \right\}^{\frac{1}{q_3}} \\ &= y^{-\frac{n}{q_3}} \left\{ \int_{\mathbb{R}^n} |F_{B_m(D_y^{p_1} f, D_y^{p_2} g)}^{yQ}(s)|^{q_3} \omega_3(y^{-1}s)^{q_3} ds \right\}^{\frac{1}{q_3}} \\ &\leq y^{-\frac{n}{q_3}} \left\{ \int_{\mathbb{R}^n} |F_{B_m(D_y^{p_1} f, D_y^{p_2} g)}^{yQ}(s)|^{q_3} (C_3(1+y^{-2}s^2)^{N_3})^{q_3} ds \right\}^{\frac{1}{q_3}}. \end{aligned}$$

We assume that  $y \leq 1$ . Then

$$\begin{aligned} \|B_{D_y^q m}(f, g)\|_{W(L^{p_3}, L_{\omega_3}^{q_3})} &\leq y^{-\frac{n}{q_3}} \left\{ \int_{\mathbb{R}^n} |F_{B_m(D_y^{p_1} f, D_y^{p_2} g)}^{yQ}(s)|^{q_3} (C_3(y^{-2} + y^{-2}s^2)^{N_3})^{q_3} ds \right\}^{\frac{1}{q_3}} \\ &= y^{-\frac{n}{q_3} - 2N_3} \|F_{B_m(D_y^{p_1} f, D_y^{p_2} g)}^{yQ}\|_{q_3, v_3}. \end{aligned}$$

Also, since  $yQ$  is a compact set, we have

$$\begin{aligned} \|B_{D_y^q m}(f, g)\|_{W(L^{p_3}, L_{\omega_3}^{q_3})} &\leq Cy^{-\frac{n}{q_3} - 2N_3} \|F_{B_m(D_y^{p_1} f, D_y^{p_2} g)}^{yQ}\|_{q_3, v_3} \\ &= Cy^{-\frac{n}{q_3} - 2N_3} \|B_m(D_y^{p_1} f, D_y^{p_2} g)\|_{W(L^{p_3}, L_{v_3}^{q_3})}. \end{aligned} \quad (2.32)$$

Since  $m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ , by Lemma 2.3 and inequality (2.32), we obtain

$$\begin{aligned} & \|B_{D_y^q m}(f, g)\|_{W(L^{p_3}, L_{\omega_3}^{q_3})} \\ & \leq C y^{-\frac{n}{q_3} - 2N_3} \|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} \|f\|_{W(L^{p_1}, L_{\omega_1}^{q_1})} \|g\|_{W(L^{p_2}, L_{\omega_2}^{q_2})}. \end{aligned} \quad (2.33)$$

Then  $D_y^q m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ , and by (2.33) we have

$$\|D_y^q m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} \leq C y^{-\frac{n}{q_3} - 2N_3} \|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))}.$$

Now let  $y > 1$ . Again, since  $m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ , by Lemma 2.3 and inequality (2.32), we obtain

$$\begin{aligned} & \|B_{D_y^q m}(f, g)\|_{W(L^{p_3}, L_{\omega_3}^{q_3})} \\ & < C \|B_m(D_y^{p_1} f, D_y^{p_2} g)\|_{W(L^{p_3}, L_{\omega_3}^{q_3})} \\ & < C y^{\frac{n}{q_1} + \frac{n}{q_2} + 2N_1 + 2N_2} \|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} \|f\|_{W(L^{p_1}, L_{\omega_1}^{q_1})} \|g\|_{W(L^{p_2}, L_{\omega_2}^{q_2})}. \end{aligned} \quad (2.34)$$

Thus  $D_y^q m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ , and by (2.34) we have

$$\|D_y^q m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} < C y^{\frac{n}{q_1} + \frac{n}{q_2} + 2N_1 + 2N_2} \|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))}. \quad \square$$

**Theorem 2.6** Let  $m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ .

(a) If  $\Phi \in L^1(\mathbb{R}^{2n})$ , then  $\Phi * m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$  and

$$\|\Phi * m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} \leq \|\Phi\|_1 \|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))}.$$

(b) If  $\Phi \in L_{\omega}^1(\mathbb{R}^{2n})$  such that  $\omega(u, v) = \omega_1(u)\omega_2(v)$ , then

$\hat{\Phi} m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$  and

$$\|\hat{\Phi} m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} \leq \|\Phi\|_{1, \omega} \|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))}.$$

*Proof* (a) Let  $f, g \in C_c^\infty(\mathbb{R}^n)$  be given. By Proposition 2.5 in [12]

$$B_{\Phi * m}(f, g)(y) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi(u, v) B_{T_{(u, v)} m}(f, g)(y) du dv.$$

If we use Theorem 2.4 and the assumption  $m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ , we have

$$\begin{aligned} & \|B_{\Phi * m}(f, g)\|_{W(L^{p_3}, L_{\omega_3}^{q_3})} \\ & \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \|\Phi(u, v) B_{T_{(u, v)} m}(f, g)\|_{W(L^{p_3}, L_{\omega_3}^{q_3})} du dv \\ & \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\Phi(u, v)| \|T_{(u, v)} m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} \|f\|_{W(L^{p_1}, L_{\omega_1}^{q_1})} \|g\|_{W(L^{p_2}, L_{\omega_2}^{q_2})} du dv \\ & = \|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} \|\Phi\|_1 \|f\|_{W(L^{p_1}, L_{\omega_1}^{q_1})} \|g\|_{W(L^{p_2}, L_{\omega_2}^{q_2})} < \infty. \end{aligned} \quad (2.35)$$

Hence  $\Phi * m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ , and by (2.35) we obtain

$$\|\Phi * m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} \leq \|\Phi\|_1 \|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))}.$$

(b) Let  $\Phi \in L^1_\omega(\mathbb{R}^{2n})$ . Take any  $f, g \in C_c^\infty(\mathbb{R}^n)$ . It is known by Proposition 2.5 in [12] that

$$B_{\hat{\Phi}m}(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi(u, v) B_{M_{(-u, -v)}m}(f, g)(x) du dv.$$

Since  $m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ , we have  $M_{(-u, -v)}m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$  and

$$\|M_{(-u, -v)}m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} \leq \omega_1(u) \omega_2(v) \|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))}$$

by Theorem 2.4. Then

$$\begin{aligned} & \|B_{\hat{\Phi}m}(f, g)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})} \\ & \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \|\Phi(u, v) B_{M_{(-u, -v)}m}(f, g)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})} du dv \\ & \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\Phi(u, v)| \|M_{(-u, -v)}m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} \|f\|_{p_1, \omega_1} \|g\|_{p_2, \omega_2} du dv \\ & \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\Phi(u, v)| \omega_1(u) \omega_2(v) \|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} \\ & \quad \times \|f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})} du dv \\ & = \|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} \|f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})} \|\Phi\|_{1, \omega}. \end{aligned} \quad (2.36)$$

Thus from (2.36) we obtain  $\hat{\Phi}m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$  and

$$\|\hat{\Phi}m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} \leq \|\Phi\|_{1, \omega} \|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))}. \quad \square$$

**Theorem 2.7** Let  $m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ . If  $Q_1, Q_2$  are bounded measurable sets in  $\mathbb{R}^n$ , then

$$\begin{aligned} h(\xi, \eta) &= \frac{1}{\mu(Q_1 \times Q_2)} \iint_{Q_1 \times Q_2} m(\xi + u, \eta + v) du dv \\ &\in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]. \end{aligned}$$

*Proof* Let  $f, g \in C_c^\infty(\mathbb{R}^n)$ . We know by Theorem 2.9 in [10] that

$$B_h(f, g)(x) = \frac{1}{\mu(Q_1 \times Q_2)} \iint_{Q_1 \times Q_2} B_{T_{(-u, -v)}m}(f, g)(x) du dv.$$

From Theorem 2.4, we have

$$\begin{aligned} & \|B_h(f, g)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})} \\ & \leq \frac{1}{\mu(Q_1 \times Q_2)} \iint_{Q_1 \times Q_2} \|B_{T_{(-u, -v)}m}(f, g)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})} du dv \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\mu(Q_1 \times Q_2)} \iint_{Q_1 \times Q_2} \|T_{(-u, -v)m}\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} \\
&\quad \times \|f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})} du dv \\
&= \frac{1}{\mu(Q_1 \times Q_2)} \|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} \\
&\quad \times \|f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})} \mu(Q_1 \times Q_2) \\
&= \|m\|_{(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))} \|f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})}.
\end{aligned}$$

Hence, we obtain  $h(\xi, \eta) \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ .  $\square$

**Theorem 2.8** Let  $\omega_3$  be a continuous, symmetric, slowly increasing weight function,  $\omega(u, v) = \omega_1(u)\omega_2(v)$ ,  $\omega_3 \leq \omega_1$ ,  $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3}$ ,  $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$ ,  $\frac{1}{p_3} + \frac{1}{p'_3} = 1$ ,  $\frac{1}{q_3} + \frac{1}{q'_3} = 1$  and  $q'_3 \geq p'_3$ . Assume that  $\Phi \in L^1_{\omega}(\mathbb{R}^{2n})$ ,  $\Psi_1 \in L^1_{\omega_1}(\mathbb{R}^n)$  and  $\Psi_2 \in L^1_{\omega_2}(\mathbb{R}^n)$ . If  $m(\xi, \eta) = \hat{\Psi}_1(\xi)\hat{\Phi}(\xi, \eta)\hat{\Psi}_2(\eta)$ , then  $m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ .

*Proof* Take any  $f, g, h \in C_c^\infty(\mathbb{R}^n)$ . Then, by Theorem 2.10 in [10], we write

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\xi + \eta) m(\xi, \eta) d\xi d\eta \right| \leq \int_{\mathbb{R}^n} |h(y) \tilde{B}_{\hat{\Phi}}(f * \Psi_1, g * \Psi_2)(y)| dy.$$

On the other hand, we know that the inequalities

$$\|f * \Psi_1\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \leq C_1 \|f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \|\Psi_1\|_{1, \omega_1} \quad (2.37)$$

and

$$\|g * \Psi_2\|_{W(L^{p_2}, L^{q_2}_{\omega_2})} \leq C_2 \|g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})} \|\Psi_2\|_{1, \omega_2} \quad (2.38)$$

hold, where  $C_1 > 0$ ,  $C_2 > 0$  by [3]. That means  $f * \Psi_1 \in W(L^{p_1}, L^{q_1}_{\omega_1})$  and  $g * \Psi_2 \in W(L^{p_2}, L^{q_2}_{\omega_2})$ . Also, every constant function is bilinear multiplier of type  $(W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3))$  under the given conditions. So, by Theorem 2.6, we have  $\hat{\Phi} \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ . Now, if we set that  $-y = u$ , we have

$$\begin{aligned}
&\|\tilde{B}_{\hat{\Phi}}(f * \Psi_1, g * \Psi_2)(y)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})} \\
&= \left\| \left\{ \int_{\mathbb{R}^n} |B_{\hat{\Phi}}(f * \Psi_1, g * \Psi_2)(u)|^{p_3} \chi_{-Q-x}(u) du \right\}^{\frac{1}{p_3}} \right\|_{q_3, \omega_3} \\
&= \|F_{B_{\hat{\Phi}}(f * \Psi_1, g * \Psi_2)}^{-Q}(-x)\|_{q_3, \omega_3}.
\end{aligned}$$

In this here, we set  $-x = u$  and use  $\omega_3$  to be symmetric. Then we have

$$\begin{aligned}
\|\tilde{B}_{\hat{\Phi}}(f * \Psi_1, g * \Psi_2)(y)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})} &\leq C_3 \|F_{B_{\hat{\Phi}}(f * \Psi_1, g * \Psi_2)}^Q\|_{q_3, \omega_3} \\
&= C_3 \|B_{\hat{\Phi}}(f * \Psi_1, g * \Psi_2)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})} \quad (2.39)
\end{aligned}$$

by [5]. Using the Hölder inequality, inequalities (2.37), (2.38), (2.39) and  $\hat{\Phi} \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ , we find

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\xi + \eta) m(\xi, \eta) d\xi d\eta \right| \\ & \leq \|h\|_{W(L^{p'_3}, L^{q'_3}_{\omega_3^{-1}})} \|B_{\hat{\Phi}}(f * \Psi_1, g * \Psi_2)\|_{W(L^{p_3}, L^{q_3}_{\omega_3})} \\ & \leq C_1 C_2 C_3 \|h\|_{W(L^{p'_3}, L^{q'_3}_{\omega_3^{-1}})} \|B_{\hat{\Phi}}\| \|\Psi_1\|_{1, \omega_1} \|\Psi_2\|_{1, \omega_2} \|f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})}. \end{aligned}$$

If we say  $C = C_1 C_2 C_3 \|B_{\hat{\Phi}}\| \|\Psi_1\|_{1, \omega_1} \|\Psi_2\|_{1, \omega_2}$ , then we obtain

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\xi + \eta) m(\xi, \eta) d\xi d\eta \right| \leq C \|f\|_{W(L^{p_1}, L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2}, L^{q_2}_{\omega_2})} \|h\|_{W(L^{p'_3}, L^{q'_3}_{\omega_3^{-1}})}.$$

From Theorem 2.2, we have  $m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ .  $\square$

**Theorem 2.9** Let  $1 \leq p_1, p_2 \leq p \leq 2$ ,  $1 \leq q_1, q_2 \leq r \leq 2$ ,  $p_3 \geq p'$ ,  $q_3 \geq r'$  and  $q'_3 \geq p'_3$  such that  $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{2}{p}$  and  $\frac{1}{q_3} = \frac{1}{q_1} + \frac{1}{q_2} - \frac{2}{r}$ . Assume that  $\omega_3$  is a continuous, bounded, symmetric weight function. If  $m \in W(L^p(\mathbb{R}^{2n}), L^r(\mathbb{R}^{2n})) \cap L^\infty(\mathbb{R}^{2n})$ , then  $m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ .

*Proof* Firstly, we show that  $m \in \text{BM}[W(p, r, \omega_1; p, r, \omega_2; \infty, \infty, \omega_3)]$ . Take any  $f, g, h \in C_c^\infty(\mathbb{R}^n)$ . Let  $A \times B \subset \mathbb{R}^{2n}$  be a closed and bounded rectangle. Since the definition of  $W(L^{p'}(\mathbb{R}^{2n}), L^{r'}(\mathbb{R}^{2n}))$  is independent of the choice of a compact set  $Q$ , then, by using Fubini's theorem, we get

$$\begin{aligned} & \|\hat{f}(\xi) \hat{g}(\eta)\|_{W(L^{p'}(\mathbb{R}^{2n}), L^{r'}(\mathbb{R}^{2n}))} \\ & = \|F_{\hat{f}\hat{g}}^Q\|_{L^{r'}(\mathbb{R}^{2n})} \leq C_1 \|F_{\hat{f}\hat{g}}^{A \times B}\|_{L^{r'}(\mathbb{R}^{2n})} \\ & = C_1 \|\hat{f} \chi_{x+A} \|_{p'} \|\hat{g} \chi_{y+B} \|_{p'} \|_{L^{r'}(\mathbb{R}^{2n})} = C_1 \|F_{\hat{f}}^A(x) F_{\hat{g}}^B(y)\|_{L^{r'}(\mathbb{R}^{2n})} \\ & = C_1 \|F_{\hat{f}}^A\|_{r'} \|F_{\hat{g}}^B\|_{r'} = C_1 \|\hat{f}\|_{W(L^{p'}, L^{r'})} \|\hat{g}\|_{W(L^{p'}, L^{r'})} \end{aligned} \quad (2.40)$$

for some  $C_1 > 0$ . So, we have  $\hat{f}(\xi) \hat{g}(\eta) \in W(L^{p'}(\mathbb{R}^{2n}), L^{r'}(\mathbb{R}^{2n}))$ . By using the Hölder inequality, the Hausdorff-Young inequality and equality (2.40), we obtain

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\xi + \eta) m(\xi, \eta) d\xi d\eta \right| \\ & \leq C_1 \|h\|_1 \|\hat{f}(\xi) \hat{g}(\eta)\|_{W(L^{p'}(\mathbb{R}^{2n}), L^{r'}(\mathbb{R}^{2n}))} \|m\|_{W(L^p(\mathbb{R}^{2n}), L^r(\mathbb{R}^{2n}))} \\ & \leq C_1 \|h\|_1 \|\hat{f}\|_{W(L^{p'}, L^{r'})} \|\hat{g}\|_{W(L^{p'}, L^{r'})} \|m\|_{W(L^p(\mathbb{R}^{2n}), L^r(\mathbb{R}^{2n}))} \\ & \leq C_1 C_2 \|h\|_{W(L^1, L^1)} \|f\|_{W(L^p, L^r)} \|g\|_{W(L^p, L^r)} \|m\|_{W(L^p(\mathbb{R}^{2n}), L^r(\mathbb{R}^{2n}))} \\ & \leq C_1 C_2 \|\omega_3\|_\infty \|m\|_{W(L^p(\mathbb{R}^{2n}), L^r(\mathbb{R}^{2n}))} \|f\|_{W(L^p, L^r_{\omega_1})} \|g\|_{W(L^p, L^r_{\omega_2})} \|h\|_{W(L^1, L^1_{\omega_3^{-1}})} \end{aligned}$$

for some  $C_2 > 0$ . Therefore  $m \in \text{BM}[W(p, r, \omega_1; p, r, \omega_2; \infty, \infty, \omega_3)]$  by Theorem 2.2.



Now, we show that  $m \in \text{BM}[W(p, r, \omega_1; 1, 1, \omega_2; p', r', \omega_3)]$ . Again, using Fubini's theorem, we have

$$\begin{aligned} & \|\hat{f}(-v)\hat{h}(u)\|_{W(L^{p'}(\mathbb{R}^{2n}), L^{r'}(\mathbb{R}^{2n}))} \\ & \leq C_3 \|\|\hat{f}(-v)\hat{h}(u)\chi_{(x,y)+A \times B}\|_{L^{p'}(\mathbb{R}^{2n})}\|_{L^{r'}(\mathbb{R}^{2n})} \\ & = C_3 \|\|\hat{f}\chi_{-x-A}\|_{p'}\|\hat{h}\chi_{y+B}\|_{p'}\|_{L^{r'}(\mathbb{R}^{2n})} = C_3 \|F_{\hat{f}}^{-A}(-x)F_{\hat{h}}^B(y)\|_{L^{r'}(\mathbb{R}^{2n})} \\ & = C_3 \|F_{\hat{f}}^{-A}\|_{r'} \|F_{\hat{h}}^B\|_{r'} \leq C_3 C_4 \|\hat{f}\|_{W(L^{p'}, L^{r'})} \|\hat{h}\|_{W(L^{p'}, L^{r'})} \end{aligned} \quad (2.41)$$

for some  $C_3 > 0$ ,  $C_4 > 0$ . So,  $\hat{f}(-v)\hat{h}(u) \in W(L^{p'}(\mathbb{R}^{2n}), L^{r'}(\mathbb{R}^{2n}))$ . Similarly, we have

$$\begin{aligned} & \|m(-v, u + v)\|_{W(L^p(\mathbb{R}^{2n}), L^r(\mathbb{R}^{2n}))} \leq C_5 \|F_m^{(-A) \times (A+B)}\|_{L^r(\mathbb{R}^{2n})} \\ & \leq C_5 C_6 \|m\|_{W(L^p(\mathbb{R}^{2n}), L^r(\mathbb{R}^{2n}))} \end{aligned} \quad (2.42)$$

for some  $C_5 > 0$ ,  $C_6 > 0$ . That means  $m(-v, u + v) \in W(L^p(\mathbb{R}^{2n}), L^r(\mathbb{R}^{2n}))$ . We set  $\xi + \eta = u$  and  $\xi = -v$ . Then, by using the Hölder inequality, the Hausdorff-Young inequality, (2.41) and (2.42), we get

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi + \eta)m(\xi, \eta) d\xi d\eta \right| \\ & = \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(-v)\hat{g}(u + v)\hat{h}(u)m(-v, u + v) du dv \right| \\ & \leq \|g\|_1 \|\hat{f}(-v)\hat{g}(u)\|_{W(L^{p'}(\mathbb{R}^{2n}), L^{r'}(\mathbb{R}^{2n}))} \|m(-v, u + v)\|_{W(L^p(\mathbb{R}^{2n}), L^r(\mathbb{R}^{2n}))} \\ & \leq C_3 C_4 C_5 C_6 \|g\|_1 \|\hat{f}\|_{W(L^{p'}, L^{r'})} \|\hat{h}\|_{W(L^{p'}, L^{r'})} \|m\|_{W(L^p(\mathbb{R}^{2n}), L^r(\mathbb{R}^{2n}))} \\ & \leq C_3 C_4 C_5 C_6 C_7 \|\omega_3\|_{\infty} \|m\|_{W(L^p(\mathbb{R}^{2n}), L^r(\mathbb{R}^{2n}))} \|f\|_{W(L^p, L_{\omega_1}^r)} \|g\|_{W(L^1, L_{\omega_2}^1)} \|h\|_{W(L^p, L_{\omega_3}^{-1})}. \end{aligned}$$

Thus, by Theorem 2.2, we obtain  $m \in \text{BM}[W(p, r, \omega_1; 1, 1, \omega_2; p', r', \omega_3)]$ . Similarly, if we change the variables  $\xi + \eta = u$  and  $\eta = -v$ , then

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi + \eta)m(\xi, \eta) d\xi d\eta \right| \\ & = \|f\|_1 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\hat{g}(-v)\hat{h}(u)| |m(u + v, -v)| du dv \\ & \leq C \|\omega_3\|_{\infty} \|m\|_{W(L^p(\mathbb{R}^{2n}), L^r(\mathbb{R}^{2n}))} \|f\|_{W(L^1, L_{\omega_1}^1)} \|g\|_{W(L^p, L_{\omega_2}^r)} \|h\|_{W(L^p, L_{\omega_3}^{-1})}. \end{aligned}$$

Hence  $m \in \text{BM}[W(1, 1, \omega_1; p, r, \omega_2; p', r', \omega_3)]$ .

We take  $\tilde{p}_1, \tilde{q}_1, \tilde{p}_3$  and  $\tilde{q}_3$  such that  $1 \leq \tilde{p}_1 \leq p$ ,  $1 \leq \tilde{q}_1 \leq r$ ,  $p' \leq \tilde{p}_3 \leq \infty$  and  $r' \leq \tilde{q}_3 \leq \infty$ . Since  $m \in \text{BM}[W(p, r, \omega_1; p, r, \omega_2; \infty, \infty, \omega_3)]$  and  $m \in \text{BM}[W(1, 1, \omega_1; p, r, \omega_2; p', r', \omega_3)]$ , we have  $m \in \text{BM}[W(\tilde{p}_1, \tilde{q}_1, \omega_1; p, r, \omega_2; \tilde{p}_3, \tilde{q}_3, \omega_3)]$  by the interpolation theorem in [13, 14] such that

$$\frac{1}{\tilde{p}_1} = \frac{1-\theta}{p} + \frac{\theta}{1}, \quad \frac{1}{\tilde{p}_3} = \frac{1-\theta}{\infty} + \frac{\theta}{p'}, \quad (2.43)$$

$$\frac{1}{\tilde{q}_1} = \frac{1-\theta}{r} + \frac{\theta}{1}, \quad \frac{1}{\tilde{q}_3} = \frac{1-\theta}{\infty} + \frac{\theta}{r'} \quad (2.44)$$

for all  $0 \leq \theta \leq 1$ . On the other hand, from equalities (2.43) and (2.44), we obtain the equalities  $\frac{1}{\tilde{p}_1} - \frac{1}{\tilde{p}_3} = \frac{1}{p}$  and  $\frac{1}{\tilde{q}_1} - \frac{1}{\tilde{q}_3} = \frac{1}{r}$ . Similarly, we take  $\tilde{p}_2, \tilde{q}_2, \tilde{r}_3$  and  $\tilde{s}_3$  such that  $1 \leq \tilde{p}_2 \leq p, 1 \leq \tilde{q}_2 \leq r, p' \leq \tilde{r}_3 \leq \infty$  and  $r' \leq \tilde{s}_3 \leq \infty$ . Again, if we use  $m \in \text{BM}[W(p, r, \omega_1; p, r, \omega_2; \infty, \infty, \omega_3)]$  and  $m \in \text{BM}[W(p, r, \omega_1; 1, 1, \omega_2; p', r', \omega_3)]$ , we have  $m \in \text{BM}[W(p, r, \omega_1; \tilde{p}_2, \tilde{q}_2, \omega_2; \tilde{r}_3, \tilde{s}_3, \omega_3)]$  by the interpolation theorem in [13, 14] such that

$$\frac{1}{\tilde{p}_2} = \frac{1-\theta}{p} + \frac{\theta}{1}, \quad \frac{1}{\tilde{r}_3} = \frac{1-\theta}{\infty} + \frac{\theta}{p'}, \quad (2.45)$$

$$\frac{1}{\tilde{q}_2} = \frac{1-\theta}{r} + \frac{\theta}{1}, \quad \frac{1}{\tilde{s}_3} = \frac{1-\theta}{\infty} + \frac{\theta}{r'} \quad (2.46)$$

for all  $0 \leq \theta \leq 1$ . So, from equalities (2.45) and (2.46), we have  $\frac{1}{\tilde{p}_2} - \frac{1}{\tilde{r}_3} = \frac{1}{p}$  and  $\frac{1}{\tilde{q}_2} - \frac{1}{\tilde{s}_3} = \frac{1}{r}$ .

Now, we choose  $\tilde{p}_1, \tilde{p}_2$  such that  $1 \leq \tilde{p}_1 \leq p_1 < p$  and  $1 \leq \tilde{p}_2 \leq p_2 < p$ . Let these numbers have the following conditions:

$$\frac{1}{\tilde{p}_1} - \frac{1}{p} = (1-\theta) \left( \frac{1}{\tilde{p}_1} - \frac{1}{p} \right), \quad (2.47)$$

$$\frac{1}{\tilde{p}_2} - \frac{1}{p} = (1-\theta) \left( \frac{1}{\tilde{p}_2} - \frac{1}{p} \right) \quad (2.48)$$

for  $0 < \theta < 1$ . Again, we choose  $\tilde{q}_1, \tilde{q}_2$  such that  $1 \leq \tilde{q}_1 \leq q_1 < r$  and  $1 \leq \tilde{q}_2 \leq q_2 < r$ . Let these numbers have the following conditions:

$$\frac{1}{\tilde{q}_1} - \frac{1}{r} = (1-\theta) \left( \frac{1}{\tilde{q}_1} - \frac{1}{r} \right), \quad (2.49)$$

$$\frac{1}{\tilde{q}_2} - \frac{1}{r} = (1-\theta) \left( \frac{1}{\tilde{q}_2} - \frac{1}{r} \right) \quad (2.50)$$

for  $0 < \theta < 1$ . If we use equalities (2.45), (2.46), (2.47), (2.48), (2.49) and (2.50), we write

$$\frac{1}{\tilde{p}_1} = \frac{1-\theta}{\tilde{p}_1} + \frac{\theta}{p}, \quad \frac{1}{\tilde{p}_2} = \frac{1-\theta}{p} + \frac{\theta}{\tilde{p}_2}, \quad (2.51)$$

$$\frac{1}{\tilde{q}_1} = \frac{1-\theta}{\tilde{q}_1} + \frac{\theta}{r}, \quad \frac{1}{\tilde{q}_2} = \frac{1-\theta}{r} + \frac{\theta}{\tilde{q}_2}. \quad (2.52)$$

Moreover, using the equalities  $\frac{1}{\tilde{p}_1} - \frac{1}{\tilde{p}_3} = \frac{1}{p}, \frac{1}{\tilde{p}_2} - \frac{1}{\tilde{r}_3} = \frac{1}{p}$ , (2.51) and the assumption  $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{2}{p}$ , we obtain

$$\frac{1}{p_3} = \frac{1-\theta}{\tilde{p}_3} + \frac{\theta}{\tilde{r}_3}. \quad (2.53)$$

Similarly, using the equalities  $\frac{1}{\tilde{q}_1} - \frac{1}{\tilde{q}_3} = \frac{1}{r}, \frac{1}{\tilde{q}_2} - \frac{1}{\tilde{s}_3} = \frac{1}{r}$ , (2.52) and the assumption  $\frac{1}{q_3} = \frac{1}{q_1} + \frac{1}{q_2} - \frac{2}{r}$ , we obtain

$$\frac{1}{q_3} = \frac{1-\theta}{\tilde{q}_3} + \frac{\theta}{\tilde{s}_3}. \quad (2.54)$$

Since  $m \in \text{BM}[\tilde{W}(\tilde{p}_1, \tilde{q}_1, \omega_1; p, r, \omega_2; \tilde{p}_3, \tilde{q}_3, \omega_3)]$ ,  $m \in \text{BM}[W(p, r, \omega_1; \tilde{p}_2, \tilde{q}_2, \omega_2; \tilde{r}_3, \tilde{s}_3, \omega_3)]$ , then the bilinear multipliers  $B_m : W(L^{\tilde{p}_1}, L^{\tilde{q}_1}_{\omega_1}) \times W(L^p, L^r_{\omega_2}) \rightarrow W(L^{\tilde{p}_3}, L^{\tilde{q}_3}_{\omega_3})$  and  $B_m :$

$W(L^p, L^r_{\omega_1}) \times W(L^{\tilde{p}_2}, L^{\tilde{q}_2}_{\omega_2}) \rightarrow W(L^{\tilde{p}_3}, L^{\tilde{q}_3}_{\omega_3})$  are bounded. Also, since  $1 \leq \tilde{p}_1 \leq p_1 < p$ ,  $1 \leq \tilde{q}_1 \leq q_1 < r$ ,  $1 \leq \tilde{q}_2 \leq q_2 < r$ ,  $1 \leq \tilde{p}_2 \leq p_2 < p$ , by equalities (2.53) and (2.54) and by the interpolation theorem in [13],  $B_m : W(L^{p_1}, L^{q_1}_{\omega_1}) \times W(L^{p_2}, L^{q_2}_{\omega_2}) \rightarrow W(L^{p_3}, L^{q_3}_{\omega_3})$  is bounded. That means  $m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$ . This completes the proof.  $\square$

### 3 The bilinear multipliers space $\text{BM}[W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3)]$

**Lemma 3.1** Let  $1 \leq p(x) \leq q < \infty$ .

- If  $\omega$  is a slowly increasing weight function, then  $C_c^\infty(\mathbb{R}^n)$  is dense in the weighted variable exponent Wiener amalgam space  $W(L^{p(x)}, L^q_\omega)$ .
- If  $\omega$  is a continuous, slowly increasing weight function, then  $C_c^\infty(\mathbb{R}^n)$  is dense in the weighted variable exponent Wiener amalgam space  $W(L^{p(x)}, L^q_{\omega^{-1}})$ .

This lemma can be proved easily by using the proof technique in Lemma 2.1.

**Definition 3.1** Let  $p_1(x), p_2(x), p_3(x) \in P(\mathbb{R}^n)$ ,  $p_1^* < \infty$ ,  $p_2^* < \infty$ ,  $p_3^* < \infty$ ,  $p_1(x) \leq q_1$ ,  $p_2(x) \leq q_2$ ,  $1 \leq q_3 \leq \infty$  and  $\omega_1, \omega_2, \omega_3$  be weight functions on  $\mathbb{R}^n$ . Assume that  $\omega_1, \omega_2$  are slowly increasing functions and  $m(\xi, \eta)$  is a bounded, measurable function on  $\mathbb{R}^n \times \mathbb{R}^n$ . Define

$$B_m(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) m(\xi, \eta) e^{2\pi i(\xi + \eta, x)} d\xi d\eta$$

for all  $f, g \in C_c^\infty(\mathbb{R}^n)$ .  $m$  is said to be a bilinear multiplier on  $\mathbb{R}^n$  of type  $(W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3))$  if there exists  $C > 0$  such that

$$\|B_m(f, g)\|_{W(L^{p_3(x)}, L^{q_3}_{\omega_3})} \leq C \|f\|_{W(L^{p_1(x)}, L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2(x)}, L^{q_2}_{\omega_2})}$$

for all  $f, g \in C_c^\infty(\mathbb{R}^n)$ . That means  $B_m$  extends to a bounded bilinear operator from  $W(L^{p_1(x)}, L^{q_1}_{\omega_1}) \times W(L^{p_2(x)}, L^{q_2}_{\omega_2})$  to  $W(L^{p_3(x)}, L^{q_3}_{\omega_3})$ . We denote by  $\text{BM}[W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3)]$  the space of all bilinear multipliers of type  $(W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3))$  and

$$\|m\|_{(W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3))} = \|B_m\|.$$

The following theorem can be proved easily by using Lemma 3.1 and the technique of proof of Theorem 2.2.

**Theorem 3.1** Let  $\frac{1}{p_3(x)} + \frac{1}{p'_3(x)} = 1$ ,  $\frac{1}{q_3} + \frac{1}{q'_3} = 1$ ,  $q'_3 \geq p'_3(x)$ ,  $p_3(-x) = p_3(x)$  and  $\omega_3$  be a continuous, symmetric, slowly increasing weight function. Then  $m \in \text{BM}[W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3)]$  if and only if there exists  $C > 0$  such that

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\xi + \eta) m(\xi, \eta) d\xi d\eta \right| \leq C \|f\|_{W(L^{p_1(x)}, L^{q_1}_{\omega_1})} \|g\|_{W(L^{p_2(x)}, L^{q_2}_{\omega_2})} \|h\|_{W(L^{p'_3(x)}, L^{q'_3}_{\omega_3^{-1}})}$$

for all  $f, g, h \in C_c^\infty(\mathbb{R}^n)$ .

Now we will give some properties of the space  $\text{BM}[W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3)]$ . Since some properties of usual Lebesgue spaces are not true in general in the vari-

able exponent Lebesgue spaces, like translation invariance, then also some properties of the spaces  $\text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3, q_3, \omega_3)]$  do not hold true in general in the spaces  $\text{BM}[W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3)]$ .

**Theorem 3.2** *If  $m \in \text{BM}[W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3)]$ , then  $T_{(\xi_0, \eta_0)} m \in \text{BM}[W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3)]$  and*

$$\|T_{(\xi_0, \eta_0)} m\|_{(W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3))} = \|m\|_{(W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3))}$$

for all  $(\xi_0, \eta_0) \in \mathbb{R}^{2n}$ .

*Proof* Let  $f, g \in C_c^\infty(\mathbb{R}^n)$ . Then we have

$$\|M_{-\xi_0} f\|_{W(L^{p_1(x)}, L_{\omega_1}^{q_1})} = \| \|e^{2\pi i(-\xi_0, \cdot)} f(\cdot) \chi_{z+Q}(\cdot) \|_{p_1(x)} \|_{q_1, \omega_1} = \|f\|_{W(L^{p_1(x)}, L_{\omega_1}^{q_1})}.$$

Similarly, the equality  $\|M_{-\eta_0} g\|_{W(L^{p_2(x)}, L_{\omega_2}^{q_2})} = \|g\|_{W(L^{p_2(x)}, L_{\omega_2}^{q_2})}$  is written. So, by using these results and the assumption  $m \in \text{BM}[W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3)]$ , we have

$$\begin{aligned} & \|B_{T_{(\xi_0, \eta_0)} m}(f, g)\|_{W(L^{p_3(x)}, L_{\omega_3}^{q_3})} \\ &= \|B_m(M_{-\xi_0} f, M_{-\eta_0} g)\|_{W(L^{p_3(x)}, L_{\omega_3}^{q_3})} \\ &\leq \|B_m\| \|M_{-\xi_0} f\|_{W(L^{p_1(x)}, L_{\omega_1}^{q_1})} \|M_{-\eta_0} g\|_{W(L^{p_2(x)}, L_{\omega_2}^{q_2})} \\ &= \|B_m\| \|f\|_{W(L^{p_1(x)}, L_{\omega_1}^{q_1})} \|g\|_{W(L^{p_2(x)}, L_{\omega_2}^{q_2})}. \end{aligned}$$

Thus  $T_{(\xi_0, \eta_0)} m \in \text{BM}[W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3)]$ . Moreover, by using the same technique as in the proof of Theorem 2.4, we obtain

$$\|T_{(\xi_0, \eta_0)} m\|_{(W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3))} = \|m\|_{(W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3))}. \quad \square$$

**Theorem 3.3** *Let  $m \in \text{BM}[W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3)]$ . Then  $\Phi * m \in \text{BM}[W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3)]$ , and there exists  $C > 0$  such that*

$$\|\Phi * m\|_{(W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3))} \leq C \|\Phi\|_1 \|m\|_{(W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3))}$$

for all  $\Phi \in L^1(\mathbb{R}^{2n})$ .

*Proof* Take any  $f, g \in C_c^\infty(\mathbb{R}^n)$ . By Proposition 2.5 in [12], we know that

$$B_{\Phi * m}(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi(u, v) B_{T_{(\xi_u, \eta_v)}} m(f, g)(x) du dv. \quad (3.1)$$

Since  $m \in \text{BM}[W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3)]$ , then by Theorem 3.2,  $T_{(u, v)} m$  in the space  $\text{BM}[W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3)]$  and

$$\|T_{(u, v)} m\|_{(W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3))} = \|m\|_{(W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3))}.$$

Using (3.1) and the Minkowski inequality in [15], we find  $C > 0$  such that

$$\begin{aligned}
 & \|B_{\Phi * m}(f, g)\|_{W(L^{p_3(x)}, L_{\omega_3}^{q_3})} \\
 &= \left\| \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi(u, v) B_{T(u, v)m}(f, g) du dv \right) \chi_{z+Q} \right\|_{p_3(x), q_3, \omega_3} \\
 &\leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\Phi(u, v)| \|B_{T(u, v)m}(f, g) \chi_{z+Q}\|_{p_3(x), q_3, \omega_3} du dv \\
 &= C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\Phi(u, v)| \|B_{T(u, v)m}(f, g)\|_{W(L^{p_3(x)}, L_{\omega_3}^{q_3})} du dv \\
 &\leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\Phi(u, v)| \|T(u, v)m\|_{\text{BM}[W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3)]} \\
 &\quad \times \|f\|_{W(L^{p_1(x)}, L_{\omega_1}^{q_1})} \|g\|_{W(L^{p_2(x)}, L_{\omega_2}^{q_2})} du dv \\
 &= C \|m\|_{\text{BM}[W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3)]} \\
 &\quad \times \|f\|_{W(L^{p_1(x)}, L_{\omega_1}^{q_1})} \|g\|_{W(L^{p_2(x)}, L_{\omega_2}^{q_2})} \|\Phi\|_1. \tag{3.2}
 \end{aligned}$$

Hence  $\Phi * m \in \text{BM}[W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3)]$ , and by (3.2) we have

$$\begin{aligned}
 & \|\Phi * m\|_{(W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3))} \\
 &\leq C \|\Phi\|_1 \|m\|_{(W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3))}. \quad \square
 \end{aligned}$$

**Theorem 3.4** Let  $\frac{1}{p_3(x)} + \frac{1}{p'_3(x)} = 1$ ,  $\frac{1}{q_3} + \frac{1}{q'_3} = 1$ ,  $q'_3 \geq p'_3(x)$ ,  $p_3(-x) = p_3(x)$  and  $\omega_3$  be a continuous, symmetric, slowly increasing weight function. If  $\Psi_1 \in L^1_{\omega_1}(\mathbb{R}^n)$ ,  $\Psi_2 \in L^1_{\omega_2}(\mathbb{R}^n)$  and  $m \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3(x), q_3, \omega_3)]$ , then  $\hat{\Psi}_1(\xi)m(\xi, \eta)\hat{\Psi}_2(\eta) \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3(x), q_3, \omega_3)]$ .

*Proof* Take any  $f, g, h \in C_c^\infty(\mathbb{R}^n)$ . Then, by Theorem 2.10 in [10], we write the following:

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\xi + \eta) m(\xi, \eta) d\xi d\eta \right| \leq \int_{\mathbb{R}^n} |h(y) \tilde{B}_m(f * \Psi_1, g * \Psi_2)(y)| dy.$$

So, by Theorem 11.7.1 in [5] and inequalities (2.37), (2.38), there exists  $C > 0$  such that

$$\begin{aligned}
 & \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\xi + \eta) \hat{\Psi}_1(\xi) m(\xi, \eta) \hat{\Psi}_2(\eta) d\xi d\eta \right| \\
 &\leq \|h\|_{W(L^{p'_3(x)}, L_{\omega_3}^{q'_3})} \|\tilde{B}_m(f * \Psi_1, g * \Psi_2)\|_{W(L^{p_3(x)}, L_{\omega_3}^{q_3})} \\
 &\leq C \|h\|_{W(L^{p'_3(x)}, L_{\omega_3}^{q'_3})} \|B_m(f * \Psi_1, g * \Psi_2)\|_{W(L^{p_3(x)}, L_{\omega_3}^{q_3})} \\
 &\leq C \|h\|_{W(L^{p'_3(x)}, L_{\omega_3}^{q'_3})} \|B_m\| \|f * \Psi_1\|_{W(L^{p_1}, L_{\omega_1}^{q_1})} \|g * \Psi_2\|_{W(L^{p_2}, L_{\omega_2}^{q_2})} \\
 &\leq C \|B_m\| \|\Psi_1\|_{1, \omega_1} \|\Psi_2\|_{1, \omega_2} \|f\|_{W(L^{p_1}, L_{\omega_1}^{q_1})} \|g\|_{W(L^{p_2}, L_{\omega_2}^{q_2})} \|h\|_{W(L^{p'_3(x)}, L_{\omega_3}^{q'_3})}.
 \end{aligned}$$

Hence,  $\hat{\Psi}_1(\xi)m(\xi, \eta)\hat{\Psi}_2(\eta) \in \text{BM}[W(p_1, q_1, \omega_1; p_2, q_2, \omega_2; p_3(x), q_3, \omega_3)]$  by Theorem 3.1.  $\square$

**Theorem 3.5** Let  $m \in \text{BM}[W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3)]$ . If  $Q_1, Q_2 \subset \mathbb{R}^n$  are bounded sets, then

$$h(\xi, \eta) = \frac{1}{\mu(Q_1 \times Q_2)} \iint_{Q_1 \times Q_2} m(\xi + u, \eta + v) du dv \\ \in \text{BM}[W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3)].$$

*Proof* Let  $f, g, h \in C_c^\infty(\mathbb{R}^n)$  be given. The equality

$$B_h(f, g)(x) = \frac{1}{\mu(Q_1 \times Q_2)} \iint_{Q_1 \times Q_2} B_{T_{(-u, -v)}m}(f, g)(x) du dv$$

is known by Theorem 2.9 in [10]. Using Theorem 3.2, there exists  $C > 0$  such that

$$\begin{aligned} & \|B_h(f, g)\|_{W(L^{p_3(x)}, L_{\omega_3}^{q_3})} \\ &= \left\| \frac{1}{\mu(Q_1 \times Q_2)} \iint_{Q_1 \times Q_2} B_{T_{(-u, -v)}m}(f, g) \chi_{z+Q} du dv \right\|_{W(L^{p_3(x)}, L_{\omega_3}^{q_3})} \\ &\leq \frac{C}{\mu(Q_1 \times Q_2)} \left\| \iint_{Q_1 \times Q_2} \|B_{T_{(-u, -v)}m}(f, g) \chi_{z+Q}\|_{p_3(x)} du dv \right\|_{q_3, \omega_3} \\ &\leq \frac{C}{\mu(Q_1 \times Q_2)} \iint_{Q_1 \times Q_2} \|B_{T_{(-u, -v)}m}(f, g) \chi_{z+Q}\|_{p_3(x)} \|du dv\|_{q_3, \omega_3} \\ &= \frac{C}{\mu(Q_1 \times Q_2)} \iint_{Q_1 \times Q_2} \|B_{T_{(-u, -v)}m}(f, g)\|_{W(L^{p_3(x)}, L_{\omega_3}^{q_3})} du dv \\ &\leq \frac{1}{\mu(Q_1 \times Q_2)} C \iint_{Q_1 \times Q_2} \|T_{(-u, -v)}m\|_{(W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3))} \\ &\quad \times \|f\|_{W(L^{p_1(x)}, L_{\omega_1}^{q_1})} \|g\|_{W(L^{p_2(x)}, L_{\omega_2}^{q_2})} du dv \\ &= C \|m\|_{(W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3))} \|f\|_{W(L^{p_1(x)}, L_{\omega_1}^{q_1})} \|g\|_{W(L^{p_2(x)}, L_{\omega_2}^{q_2})}. \end{aligned}$$

Hence  $h(\xi, \eta) \in m \in \text{BM}[W(p_1(x), q_1, \omega_1; p_2(x), q_2, \omega_2; p_3(x), q_3, \omega_3)]$ .  $\square$

**Theorem 3.6** Let  $r(x) \leq s(x)$ ,  $m(x) \leq q(x)$ ,  $n(x) \leq p(x)$ ,  $s \leq r$ ,  $q \leq m$ ,  $p \leq n$ ,  $v_3 \leq \omega_3$ ,  $\omega_2 \leq v_2$ ,  $\omega_1 \leq v_1$ . Then

$$\begin{aligned} & \text{BM}[W(n(x), n, \omega_1; m(x), m, \omega_2; s(x), s, \omega_3)] \\ & \subset \text{BM}[W(p(x), p, v_1; q(x), q, v_2; r(x), r, v_3)]. \end{aligned}$$

*Proof* Take any  $m \in \text{BM}[W(n(x), n, \omega_1; m(x), m, \omega_2; s(x), s, \omega_3)]$ . Then there exists  $C_1 > 0$  such that

$$\|B_m(f, g)\|_{W(L^{s(x)}, L_{\omega_3}^s)} \leq C_1 \|f\|_{W(L^{n(x)}, L_{\omega_1}^n)} \|g\|_{W(L^{m(x)}, L_{\omega_2}^m)}. \quad (3.3)$$

On the other hand, by Proposition 2.5 in [7] we have  $W(L^{s(x)}, L_{\omega_3}^s) \subset W(L^{r(x)}, L_{v_3}^r)$ ,  $W(L^{p(x)}, L_{v_1}^p) \subset W(L^{n(x)}, L_{\omega_1}^n)$  and  $W(L^{q(x)}, L_{v_2}^q) \subset W(L^{m(x)}, L_{\omega_2}^m)$ . So, there exist  $C_2 > 0$ ,

$C_3 > 0$  and  $C_4 > 0$  such that

$$\|B_m(f, g)\|_{W(L^{r(x)}, L_{v_3}^r)} \leq C_2 \|B_m(f, g)\|_{W(L^{s(x)}, L_{\omega_3}^s)}, \quad (3.4)$$

$$\|f\|_{W(L^{n(x)}, L_{\omega_1}^n)} \leq C_3 \|f\|_{W(L^{p(x)}, L_{v_1}^p)} \quad (3.5)$$

and

$$\|g\|_{W(L^{m(x)}, L_{\omega_2}^m)} \leq C_4 \|g\|_{W(L^{q(x)}, L_{v_2}^q)}. \quad (3.6)$$

Combining (3.3), (3.4), (3.5) and (3.6), we get

$$\|B_m(f, g)\|_{W(L^{r(x)}, L_{v_3}^r)} \leq C_1 C_2 C_3 C_4 \|f\|_{W(L^{p(x)}, L_{v_1}^p)} \|g\|_{W(L^{q(x)}, L_{v_2}^q)}.$$

That means  $m \in \text{BM}[W(p(x), p, v_1; q(x), q, v_2; r(x), r, v_3)]$ . Hence, we obtain  $\text{BM}[W(n(x), n, \omega_1; m(x), m, \omega_2; s(x), s, \omega_3)] \subset \text{BM}[W(p(x), p, v_1; q(x), q, v_2; r(x), r, v_3)]$ .  $\square$

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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